The Mathematics of Harmony: Clarifying the Origins and Development of Mathematics

Alexey Stakhov[1]

The International Club of the Golden Section
6 McCreary Trail, Bolton, ON, L7E 2C8, Canada
goldenmuseum@rogers.com • www.goldenmuseum.com

Abstract

This study develops a new approach to the history of mathematics. We analyze "strategic mistakes" in the development of mathematics and mathematical education (including severance of the relationship between mathematics and the theoretical natural sciences, neglect of the "golden section," the one-sided interpretation of Euclid's *Elements*, and the distorted approach to the origins of mathematics). We develop the "Mathematics of Harmony" as a new interdisciplinary direction for modern science by applying to it Dirac's "Principle of Mathematical Beauty" and discussing its role in overcoming these "strategic mistakes."

1. Introduction

1.1. *Dirac's Principle of Mathematical Beauty*

Recently the author studied the contents of a public lecture: **"The complexity of finite sequences of zeros and units, and the geometry of finite functional spaces"** [1] by eminent Russian mathematician and academician Vladimir Arnold, presented before the Moscow Mathematical Society on May 13, 2006. Let us consider some of its general ideas. Arnold notes:

1. *In my opinion, mathematics is simply a part of physics, that is, it is an experimental science, which discovers for mankind the most important and simple laws of nature.*

2. *We must begin with a beautiful mathematical theory. Dirac states:* ***"If this theory is really beautiful, then it necessarily will appear as a fine model of important physical phenomena. It is necessary to search for these phenomena to develop applications of the beautiful mathematical theory and to interpret them as predictions of new laws of physics."*** *Thus, according to Dirac, all new physics, including relativistic and quantum, develop in this way.*

At Moscow University there is a tradition that the distinguished visiting-scientists are requested to write on a blackboard a self-chosen inscription. When Dirac visited Moscow in 1956, he wrote *"A physical law must possess mathematical beauty."* This inscription is the famous *Principle of Mathematical Beauty* that Dirac developed during his scientific life. No other modern physicist has been preoccupied with the concept of beauty more than Dirac.

Thus, according to Dirac, the **Principle of Mathematical Beauty** is the primary criterion for a mathematical theory to be considered as a model of physical phenomena. Of course, there is an element of subjectivity in the definition of the "beauty" of mathematics, but the majority of mathematicians agrees that "beauty" in mathematical objects and theories nevertheless exist.

[1]**Academician of the International Higher Education Academy of Sciences**

Let's examine some which have a direct relation to the theme of this paper.

1.2. *Binomial coefficients, the binomial formula, and Pascal's triangle*

For the given non-negative integers n and k, there is the following beautiful formula that sets the *binomial coefficients*:

$$C_n^k = \frac{n!}{k!(n-k)!}, \quad (1)$$

where $n!=1\times 2\times 3\times\ldots\times n$ is a *factorial* of n.

One of the most beautiful mathematical formulas, the *binomial formula,* is based upon the binomial coefficients:

$$(a+b)^n = a^n + C_n^1 a^{n-1}b + C_n^2 a^{n-2}b^2 + \ldots + C_n^k a^{n-k}b^k + \ldots + C_n^{n-1} ab^{n-1} + b^n. \quad (2)$$

There is a very simple method for calculation of the binomial coefficients based on their following graceful properties called *Pascal's rule*:

$$C_{n+1}^k = C_n^{n-1} + C_n^k. \quad (3)$$

Using the recursive relation (3) and taking into consideration that $C_n^0 = C_n^n = 1$ and $C_n^k = C_n^{n-k}$, we can construct the following beautiful table of binomial coefficients called *Pascal's triangle* (see Table 1).

Table 1. Pascal's triangle

									1									
								1		1								
							1		2		1							
						1		3		3		1						
					1		4		6		4		1					
				1		5		10		10		5		1				
			1		6		15		20		15		6		1			
		1		7		21		35		35		21		7		1		
	1		8		28		56		70		56		28		8		1	
1		9		36		84		126		126		84		36		9		1

Here we attribute "beautiful" to all the mathematical objects above. They are widely used in both mathematics and physics.

1.3. *Fibonacci and Lucas numbers, the Golden Mean and Binet Formulas*

Let us consider the simplest recursive relation:

$$F_n=F_{n-1}+F_{n-2}, \quad (4)$$

where $n=0,\pm1,\pm2,\pm3,\ldots$. This recursive relation was introduced for the first time by the famous Italian mathematician Leonardo of Pisa (nicknamed *Fibonacci*). For the seeds

$$F_0=0 \text{ and } F_1=1 \quad (5)$$

the recursive relation (5) generates a numerical sequence called the *Fibonacci numbers* (see Table 2).

In the 19th century the French mathematician **Francois Edouard Anatole Lucas** (1842-1891) introduced the so-called *Lucas numbers* (see Table 2) given by the recursive relation

$$L_n=L_{n-1}+L_{n-2} \quad (6)$$

with the seeds

$$L_0=2 \text{ and } L_1=1. \quad (7)$$

Table 2. Fibonacci and Lucas numbers

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	-1	2	-3	5	-8	13	-21	34	-55
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	-1	3	-4	7	-11	18	-29	47	-76	123

It follows from Table 2 that the Fibonacci and Lucas numbers build up two infinite numerical sequences, each possessing graceful mathematical properties. As can be seen from Table 2, for the odd indices n=2k+1 the elements F_n and F_{-n} of the Fibonacci sequence coincide, that is, $F_{2k+1}= F_{-2k-1}$, and for the even indices n=2k they are opposite in sign, that is, F_{2k}=F_{-2k}. For the Lucas numbers L_n all is vice versa, that is, L_{2k}=L_{-2k}; L_{2k+1}= L_{-2k-1}.

In the 17th century the famous astronomer **Giovanni Domenico Cassini** (1625-1712) deduced the following beautiful formula, which connects three adjacent Fibonacci numbers in the Fibonacci sequence:

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}. \tag{8}$$

This wonderful formula evokes a reverent thrill, if one recognizes that it is valid for any value of n (n can be any integer within the limits of -∞ to +∞). The alternation of +1 and -1 in the expression (8) within the succession of all Fibonacci numbers results in the experience of genuine aesthetic enjoyment of its rhythm and beauty.

If we take the ratio of two adjacent Fibonacci numbers F_n/F_{n-1} and direct this ratio towards infinity, we arrive at the following unexpected result:

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \tau = \frac{1+\sqrt{5}}{2}, \tag{9}$$

where τ is the famous irrational number, which is the positive root of the algebraic equation:

$$x^2=x+1. \tag{10}$$

The number τ has many beautiful names – *the golden section*, *golden number, golden mean, golden proportion, and the divine proportion*. See Olsen page 2 [2].

Note that formula (9) is sometimes called *Kepler's formula* after **Johannes Kepler** (1571-1630) who deduced it for the first time.

In the 19th century French mathematician **Jacques Philippe Marie Binet (**1786-1856**)** deduced the two magnificent *Binet formulas*:

$$F_n = \frac{\tau^n - (-1)^n \tau^{-n}}{\sqrt{5}} \tag{11}$$

$$L_n = \tau^n + (-1)^n \tau^{-n} \tag{12}$$

The golden section or *division of a line segment in extreme and mean ratio* descended to us from Euclid's *Elements.* Over the many centuries the golden mean has been the subject of enthusiastic worship by outstanding scientists and thinkers including Pythagoras, Plato, Leonardo da Vinci, Luca Pacioli, Johannes Kepler and several others. In this connection, we should recall Kepler's saying concerning the golden section:

"Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first, we may compare to a measure of gold; the second we may name a precious stone."

Alexey Losev, the Russian philosopher and researcher into the aesthetics of Ancient Greece and the Renaissance, expressed his delight in the golden section and Plato's cosmology in the following words:

"From Plato's point of view, and generally from the point of view of all antique cosmology, the universe is a certain proportional whole that is subordinated to the law of harmonious division, the Golden Section... This system of cosmic proportions is sometimes considered by literary critics as a curious result of unrestrained and preposterous fantasy. Total anti-scientific weakness resounds in the explanations of those who declare this. However, we can understand this historical and aesthetic phenomenon only in conjunction with an integral comprehension of history, that is, by employing a dialectical and materialistic approach to culture and by searching for the answer in the peculiarities of ancient social existence."

We can ask the question: in what way is the "golden mean" reflected in contemporary mathematics? Unfortunately, the answer forced upon us is - only in the most impoverished manner. In mathematics, Pythagoras and Plato's ideas are considered to be a "curious result of unrestrained and preposterous fantasy." Therefore, the majority of mathematicians consider study of the golden section as a mere pastime, which is unworthy of the serious mathematician. Unfortunately, we can also find neglect of the golden section in contemporary theoretical physics. In 2006 "BINOM" publishing house (Moscow) published the interesting scientific book *Metaphysics: Century XXI* [3]. In the Preface to the book, its compiler and editor Professor Vladimirov (Moscow University) wrote:

"The third part of this book is devoted to a discussion of numerous examples of the manifestation of the 'golden section' in art, biology and our surrounding reality. However, paradoxically, the 'golden proportion' is not reflected in contemporary theoretical physics. In order to be convinced of this fact, it is enough to merely browse 10 volumes of Theoretical Physics by Landau and Lifshitz. The time has come to fill this gap in physics, all the more given that the "golden proportion" is closely connected with metaphysics and 'trinitarity' [the 'triune' nature of things]."

During several decades, the author has developed a new mathematical direction called *The Mathematics of Harmony* [4-38]. For the first time, the name of *The Harmony of Mathematics* was introduced by the author in 1996 in the lecture, *The Golden Section and Modern Harmony Mathematics* [14], presented at the session of the 7th International conference *Fibonacci Numbers and Their Applications* (Austria, Graz, July 1996). The book *The Mathematics of Harmony: from Euclid to Contemporary Mathematics and Computer Science* [11] was accepted for publication by the international publisher "World Scientific" as the fulfillment of the author's research in this area.

This present article has three goals:

1. To analyze the "strategic mistakes" in the mathematical development and demonstrate the role of the Mathematics of Harmony in the general development of the mathematics.
2. To examine the basic theories of the Mathematics of Harmony from the point of view of Dirac's *Principle of Mathematical Beauty*.
3. To demonstrate applications of the Mathematics of Harmony in modern science.

2. The "Strategic mistakes" in the development of mathematics

2.1. *Mathematics: The Loss of Certainty.* The book *Mathematics: The Loss of Certainty* [39] by Morris Kline is devoted to an analysis of the crisis of 20th century mathematics. Kline wrote:

"The history of mathematics is crowned with glorious achievements but also a record of calamities. The loss of truth is certainly a tragedy of the first magnitude, for truths are man's dearest possessions and a loss of even one is cause for grief. The realization that the splendid showcase of human reasoning exhibits a by no means perfect structure but one marred by shortcomings and vulnerable to the discovery of disastrous contradictions at any time is another blow to the stature of mathematics. But these are not the only grounds for distress. Grave misgivings and cause for dissension among mathematicians stem from the direction which research of the past one hundred years has taken. Most mathematicians have withdrawn from the world to concentrate on problems generated within mathematics. They have abandoned science. This change in direction is often described as the turn to pure as opposed to applied mathematics."

Further we read:

"Science had been the life blood and sustenance of mathematics. Mathematicians were willing partners with physicists, astronomers, chemists, and engineers in the scientific enterprise. In fact, during the 17th and 18th centuries and most of the 19th, the distinction between mathematics and theoretical science was rarely noted. And many of the leading mathematicians did far greater work in astronomy, mechanics, hydrodynamics, electricity, magnetism, and elasticity than they did in mathematics proper. Mathematics was simultaneously the queen and the handmaiden of the sciences."

Kline notes that our great predecessors were not interested in the problems of "pure mathematics," which were put forward in the forefront of the 20th century mathematics. In this connection, Kline writes:

"However, pure mathematics totally unrelated to science was not the main concern. It was a hobby, a diversion from the far more vital and intriguing problems posed by the sciences. Though Fermat was the founder of the theory of numbers, he devoted most of his efforts to the creation of analytic geometry, to problems of the calculus, and to optics .... He tried to interest Pascal and Huygens in the theory of numbers but failed. Very few men of the 17th century took any interest in that subject." Felix Klein, who was the recognized head of the mathematical world at the boundary of the 19th and 20th centuries, considered it necessary to protest against striving for abstract, "pure" mathematics:

"We cannot help feeling that in the rapid developments of modern thought, our science is in danger of becoming more and more isolated. The intimate mutual relation between mathematics and theoretical natural science which, to the lasting benefit of both sides, existed ever since the rise of modern analysis, threatens to be disrupted."

Richard Courant, who headed the Institute of Mathematical Sciences of New York University, also treated disapprovingly the passion for "pure" mathematics. He wrote in 1939:

"A serious threat to the very life of science is implied in the assertion that mathematics is nothing but a system of conclusions drawn from the definition and postulates that must be consistent but otherwise may be created by the free will of mathematicians. If this description were accurate, mathematics could not attract any intelligent person. It would be a game with definitions, rules, and syllogisms without motivation or goal. The notion that the intellect can create meaningful postulational systems at its whim is a deceptive half-truth. Only under the discipline of responsibility to the organic whole, only guided by intrinsic necessity, can the free mind achieve results of scientific value."

At present, mathematicians turned their attention to the solution of old mathematical problems formulated by the great mathematicians of the past. *Fermat's Last Theorem* is one of them. This theorem can be formulated very simply. Let us prove that for $n>2$ any integers x, y, z do not satisfy the correlation $x^n + y^n = z^n$. The theorem was formulated by Fermat in 1637 in the margins of Diophantus of Alexandria's book *Arithmetica* along with a postscript that the witty proof he found was too long to be placed there. Over the years many outstanding mathematicians (including Euler, Dirichlet, Legandre and others) tried to solve this problem. The proof of Fermat's Last Theorem was completed in 1993 by Andrew Wiles, a British mathematician working in the United States at Princeton University. The proof required 130 pages in the *Annals of Mathematics.*

Gauss was a recognized specialist in number theory, confirmed by the publication of his book *Arithmetical Researchers* (1801). In this connection, it is curious to find Gauss' opinion about *Fermat's Last Theorem*. Gauss explained in one of his letters why he did not study Fermat's problem. From his point of view, **"Fermat's hypothesis is an isolated theorem, connected with nothing, and therefore this theorem holds no interest" [39].** We should not forget that Gauss treated with great interest all 19th century mathematical problems and discoveries. In particular, Gauss was the first mathematician who supported Lobachevski's researchers on Non-Euclidean geometry. Without a doubt, Gauss' opinion about *Fermat's Last Theorem* somewhat diminishes Wiles' proof of this theorem. In this connection, we can ask the following questions: (1) What significance does Fermat's Last Theorem hold for the development of modern science? (2) Can we compare the solution of Fermat's problem with the discovery of Non-Euclidean geometry in the first half of the 19th century and other mathematical discoveries? (3) Is Fermat's Last Theorem an "aimless play of intellect" and its proof merely a demonstration of the imaginative power of human intellect - and nothing more?

Thus, following Felix Klein, Richard Courant and other famous mathematicians, Morris Kline asserted that **the main reason for the contemporary crisis in mathematics was the severance of the relationship between mathematics and theoretical natural sciences that is the greatest "strategic mistake" of 20th century mathematics.**

2.2. ***The neglect of the "beginnings."*** Eminent Russian mathematician Kolmogorov wrote a preface to the Russian translation of Lebegue's book *About the Measurement of Magnitudes* [40]. He stated that *"there is a tendency among mathematicians to be ashamed of the origin of mathematics. In comparison with the crystal clarity of the theory of its development it seems unsavory and an unpleasant pastime to rummage through the origins of its basic notions and assumptions. All building up of school algebra and all mathematical analysis might be constructed on the notion of real number without any mention of the measurement of specific magnitudes (lengths, areas, time intervals, and so on). Therefore, one and the same tendency shows itself at different stages of education and with different degrees of inclination to introduce numbers possibly sooner, and furthermore to speak only about numbers and relations between them. Lebegue protests against this tendency!"*

In this statement, Kolmogorov recognized a peculiarity of mathematicians - the diffident attitude towards the "origins" of mathematics. However, long before Kolmogorov, Nikolay Lobachevski also recognized this tendency:

"Algebra and Geometry have one and the same fate. Their very slow successes followed after the fast ones at the beginning. They left science in a state very far from perfect. It probably happened, because mathematicians turned all their attention towards the advanced aspects of analytics, and have neglected the origins of mathematics by being unwilling to dig in the field already harvested by them and now left behind."

However, just as Lobachevski demonstrated by his research that the "origins" of mathematical sciences, in particular, Euclid's *Elements* are an inexhaustible source of new mathematical ideas and discoveries. *Geometric Researches on Parallel Lines* (1840) by Lobachevski opens with the following words:

"I have found some disadvantages in geometry, reasons why this science did not until now step beyond the bounds *of Euclid's Elements. We are talking here about the first notions surrounding geometric magnitudes, measurement methods, and finally, the important gap in the theory of parallel lines ...."*

Thankfully, Lobachevski, unlike other mathematicians did not neglect concern with “origins.” His thorough analysis of the *Fifth Euclidean Postulate* (“the important gap in the theory of parallel lines”) led him to the creation of Non-Euclidean geometry – the most important mathematical discovery of the 19th century.

2.3. *The neglect of the Golden Section.* Pythagoreans advanced for the first time the brilliant idea about the harmonic structure of the Universe, including not only nature and people, but also everything in the entire cosmos. According to the Pythagoreans, “harmony is an inner connection of things without which the cosmos cannot exist.” At last, according to Pythagoras, harmony had numerical expression, that is, it is connected with the concept of number. Aristotle noticed in his Metaphysics just this peculiarity of the Pythagorean doctrine:

“The so-called Pythagoreans, who were the first to take up mathematics, not only advanced this study, but also having been brought up in it they thought its principles were the principles of all things ... since, then, all other things seemed in their whole nature to be modeled on numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole cosmos to be a harmony and a number.”

The Pythagoreans recognized that the shape of the Universe should be harmonious and all its “elements” connected with harmonious figures. Pythagoras taught that the Earth arose from cube, Fire from pyramid (tetrahedron), Air from octahedron, Water from icosahedron, the sphere of the Cosmos (the ether) from dodecahedron.

The famous Pythagorean doctrine of the “harmony of spheres” is of course connected with the harmony concept. Pythagoras and his followers held that the movement of heavenly bodies around the central world fire creates a wonderful music, which is perceived not by ear, but by intellect. The doctrine about the “harmony of the spheres,” the unity of the microcosm and macrocosm, and the doctrine about proportions - unified together provide the basis of the Pythagorean doctrine.

The main conclusion, following from Pythagorean doctrine, is that harmony is objective; it exists independently from our consciousness and is expressed in the harmonious structure of the Universe from the macrocosm down to the microcosm. However, if harmony is in fact objective, it should become a central subject of mathematical research.

The Pythagorean doctrine of numerical harmony in the Universe influenced the development of all subsequent doctrines about nature and the essence of aesthetics. This brilliant doctrine was reflected and developed in the works of great thinkers, in particular, in Plato’s cosmology. In his works, Plato developed Pythagorean doctrine and especially emphasized the cosmic significance of harmony. He was firmly convinced that harmony can be expressed by numerical proportions. This Pythagorean influence was traced especially in his *Timaeus*, where Plato, after Pythagoras, developed a doctrine about proportions and analyzed the role of the regular polyhedra (Platonic Solids), which, in his opinion, underlie the Universe itself.

The “golden section,” which was called in that period the “division in extreme and mean ratio,” played a special role in ancient science, including Plato’s cosmology. Above we presented Kepler’s and Losev’s statements about the role of the golden section in geometry and Greek culture. Kepler’s assertion raises the significance of the golden section up to the level of the *Pythagorean Theorem* - one of the most famous theorems of geometry. As a result of the unilateral approach to mathematical education each school-child knows the Pythagorean Theorem, but has a rather vague concept of the golden section - the second “treasure of geometry.” The majority of school textbooks on geometry go back in their origin to Euclid’s *Elements*. But then we may ask the question: why in the majority of them is there no real significant mention of the golden section, described for the first time in Euclid’s *Elements*? The impression created is that “the materialistic pedagogy” have thrown out the golden section from mathematical education on to the dump heap of "doubtful scientific concepts” together with astrology and other so-called esoteric sciences (where the golden section is widely emphasized). We consider this sad fact to be one of the "strategic mistakes" of modern mathematical education.

Many mathematicians interpret Kepler's statement as a great overstatement regarding the golden section. However, we should not forget that Kepler was not only a brilliant astronomer, but also a great physicist and great mathematician (in contrast to the mathematicians who criticize Kepler). In his first book *Mysterium Cosmographicum* (*The Cosmographic Mystery*), Kepler created an original model of the Solar System based on the *Platonic Solids*. He was one of the first scientists, who started to study the "Harmony of the Universe" in his book *Harmonices Mundi* (*Harmony of the World*). In *Harmony*, he attempted to explain the proportions of the natural world – particularly the astronomical and astrological aspects –in terms of music. The *Musica Universalis* or *Music of the Spheres*, studied by Ptolemy and many others before Kepler, was his main idea. From there, he extended his harmonic analysis to music, meteorology and astrology; harmony resulted from the tones made by the souls of heavenly bodies – and in the case of astrology, the interaction between those tones and human souls. In the final portion of the work (Book V), Kepler dealt with planetary motions, especially relationships between orbital velocity and orbital distance from the Sun. Similar relationships had been used by other astronomers, but Kepler – with Tycho's data and his own astronomical theories – treated them much more precisely and attached new physical significance to them.

Thus, the neglect of the "golden section" and its associated "idea of harmony" is one more "strategic mistake" in not only mathematics and mathematical education, but also theoretical physics. This mistake resulted in a number of other "strategic mistakes" in the development of mathematics and mathematical education.

2.4. *The one-sided interpretation of Euclid's Elements.* Euclid's *Elements* is the primary work of Greek mathematics. It is devoted to the axiomatic construction of geometry, and led to the "axiomatic approach" widely used in mathematics. This view of the *Elements* is widespread in contemporary mathematics. In his *Elements* Euclid collected and logically analyzed all achievements of the previous period in the field of geometry. At the same time, he presented the basis of number theory. For the first time, Euclid proved the infinity of *prime numbers* and constructed a full theory of divisibility. At last, in Books II, VI and X, we find the description of a so-called geometrical algebra that allowed Euclid to not only solve quadratic equations, but also perform complex transformations on quadratic irrationals.

Euclid's *Elements* fundamentally influenced mathematical education. Without exaggeration it is reasonable to suggest, that the contents of mathematical education in modern schools is on the whole based upon the mathematical knowledge presented in Euclid's *Elements*.

However, there is another point of view on Euclid's *Elements* suggested by **Proclus Diadochus** (412-485), the best commentator on Euclid's *Elements*. The final book of Euclid's *Elements,* Book XIII*,* is devoted to a description of the theory of the five regular polyhedra that played a predominate role in Plato's cosmology. They are well known in modern science under the name *Platonic Solids*. Proclus did pay special attention to this fact. As is generally the case, the most important data are presented in the final part of a scientific book. Based on this fact, Proclus asserts that **Euclid created his *Elements* primarily not to present an axiomatic approach to geometry, but in order to give a systematic theory of the construction of the 5 Platonic Solids, in passing throwing light on some of the most important achievements of Greek mathematics.** Thus, "Proclus' hypothesis" allows one to suppose that it was well-known in ancient science that the "Pythagorean Doctrine about the Numerical Harmony of the Cosmos" and "Plato's Cosmology," based on the regular polyhedra, were embodied in Euclid's *Elements,* the greatest Greek work of mathematics. From this point of view, **we can interpret Euclid's *Elements* as the first attempt to create a "Mathematical Theory of Harmony" which was the primary idea in Greek science.**

This hypothesis is confirmed by the geometric theorems in Euclid's *Elements*. **The problem of division in extreme and mean ratio** described in Theorem II.11 is one of them. This division named later the golden section was used by Euclid for the geometric construction of the isosceles triangle with the angles 72°, 72° & 36° (the "golden" isosceles triangle) and then of the regular pentagon and dodecahedron. We ascertain with great regret that "Proclus' hypothesis" was not really recognized by modern mathematicians who continue to consider the axiomatic statement of geometry as the main achievement of Euclid's *Elements*. However, as Euclid's *Elements* are the beginnings of school mathematical education, we should ask the question: why do the golden section and Platonic Solids occupy such a modest place in modern mathematical education?

The narrow one-sided interpretation of Euclid's *Elements* is one more "strategic mistake" in the development of mathematics and mathematical education. This "strategic mistake" resulted in a distorted picture of the history of mathematics.

2.5. *The one-sided approach to the origin of mathematics.* The traditional approach to the origin of mathematics consists of the following [41]. Historically, two practical problems stimulated the development of mathematics on in its earlier stages of development. We are referring to the **"count problem"** and **"measurement problem."** The "count problem" resulted in the creation of the first methods of number representation and the first rules for the fulfillment of arithmetical operations (including the Babylonian sexagecimal number system, Egyptian decimal arithmetic). The formation of the concept of **natural number** was the main result of this long period in the mathematics history. On the other hand, the "measurement problem" underlies the creation of geometry ("Measurement of the Earth"). The discovery of **incommensurable line segments** is considered to be the major mathematical discovery in this field. This discovery resulted in the introduction of **irrational numbers,** the next fundamental notion of mathematics following natural numbers**.**

The concepts of **natural number** and **irrational number** are the major fundamental mathematical concepts, without which it is impossible to imagine the existence of mathematics. These concepts underlie "Classical Mathematics."

Neglect of the "harmony problem" and "golden section" by mathematicians has an unfortunate influence on the development of mathematics and mathematical education. As a result, we have a one-sided view of the origin of mathematics which is one more "strategic mistake" in the development of mathematics and mathematical education.

2.6. *The underestimation of Binet formulas.* In the 19th century a theory of the "golden section" and Fibonacci numbers was supplemented by one important result. This was with the so-called **Binet formulas** for Fibonacci and Lucas numbers given by (11) and (12).

The analysis of the Binet formulas (11) and (12) gives one the opportunity to sense the beauty of mathematics and once again be convinced of the power of the human intellect! Actually, we know that the Fibonacci and Lucas numbers are always integers. But any power of the golden mean is an irrational number. As it follows from the Binet formulas, the integer numbers F_n and L_n can be represented as the difference or sum of irrational numbers, namely the powers of the golden mean! We know it is not easy to explain to pupils the concept of irrationals. For learning mathematics, the Binet formulas (10) and (11), which connect Fibonacci and Lucas numbers with the golden mean τ, are very important because they demonstrate visually a connection between integers and irrational numbers.

Unfortunately, in classical mathematics and mathematical education the Binet formulas did not get the proper kind of recognition as did, for example, "Euler formulas" and other famous mathematical formulas. Apparently, this attitude towards the Binet formulas is connected with the golden mean, which always provoked an “allergic reaction” in many mathematicians. Therefore, the Binet formulas are not generally found in school mathematics textbooks.

However, the main “strategic mistake” in the underestimation of Binet formulas is the fact that mathematicians could not see in Binet formulas a prototype for a new class of hyperbolic functions – the hyperbolic Fibonacci and Lucas functions. Such functions were discovered roughly 100 years later by Ukrainian researchers Bodnar [42], Stakhov, Tkachenko, and Rozin [9, 13, 20, 29, 33]. If the hyperbolic functions on Fibonacci and Lucas had been discovered in the 19th century, hyperbolic geometry and its applications to theoretical physics would have received a new impulse in their development.

2.7. ***The underestimation of Felix Klein's idea concerning the Regular Icosahedron.*** The name Felix Klein is well known in mathematics. In the 19th century Felix Klein tried to unite all branches of mathematics on the base of the regular icosahedron dual to the dodecahedron [43].

Klein interprets the regular icosahedron based on the “golden section” as a geometric object, connected with 5 mathematical theories: *Geometry, Galois Theory, Group Theory, Invariant Theory,* and *Differential Equations.* Klein’s main idea is extremely simple: *“Each unique geometric object is connected one way or another with the properties of the regular icosahedron.”* **Unfortunately, this remarkable idea was not developed in contemporary mathematics, which is one more “strategic mistake” in the development of mathematics.**

2.8. ***The underestimation of Bergman’s number system.*** One “strange” tradition exists in mathematics. It is usually the case that mathematicians underestimate the mathematical achievements of their contemporaries. The epochal mathematical discoveries, as a rule, in the beginning go unrecognized by mathematicians. Sometimes they are subjected to sharp criticism and even to gibes. Only after approximately 50 years, as a rule, after the death of the authors of the outstanding mathematical discoveries, the new mathematical theories are recognized and take their place of worth in mathematics. The dramatic destinies of Lobachevski, Abel, and Galois are very well-known.

In 1957 the American mathematician George Bergman published the article *A number system with an irrational base* [44]. In this article Bergman developed a very unusual extension of the notion of the positional number system. He suggested that one use the golden mean $\tau=\frac{1+\sqrt{5}}{2}$ as the basis of a special positional number system. If we use the sequences τ^i $\{i=0, \pm1, \pm2, \pm3, \ldots\}$ as “digit weights” of the “binary” number system, we get the “binary” number system with irrational base τ:

$$A=\sum_i a_i\tau^i \tag{13}$$

where A is a real number, a_i are binary numerals 0 or 1, $i = 0, \pm 1, \pm 2, \pm 3 \ldots$, τ^i is the weight of the i-th digit, τ is the base of the number system (13).

Unfortunately, Bergman’s article [44] was not noticed by mathematicians of that period. Only the journalists were surprised by the fact that George Bergman made his mathematical discovery at the age of 12! In this connection, TIME Magazine published an article about mathematical talent in America. In 50 years, according to "mathematical tradition" the time had come to evaluate the role of Bergman’s system for the development of contemporary mathematics.

The “strategic” importance of Bergman’s system is the fact that **it overturns our ideas about positional number systems, moreover, our ideas about correlations between rational and irrational numbers.**

As is well known, historically natural numbers were first introduced, after them rational numbers as ratios of natural numbers, and later – after the discovery of the "incommensurable line segments" - irrational numbers, which cannot be expressed as ratios of natural numbers. By using the traditional positional number systems (binary, ternary, decimal and so on), we can represent any natural, real or irrational number by using number systems with a base of (2, 3, 10 and so on). The base in Bergman's system [44] is the golden mean. By using Bergman's system (13), we can represent all natural, real and irrational numbers. As Bergman's system (13) is fundamentally a new positional number system, its study is very important for school mathematical education because it expands our ideas about the positional principle of number representation.

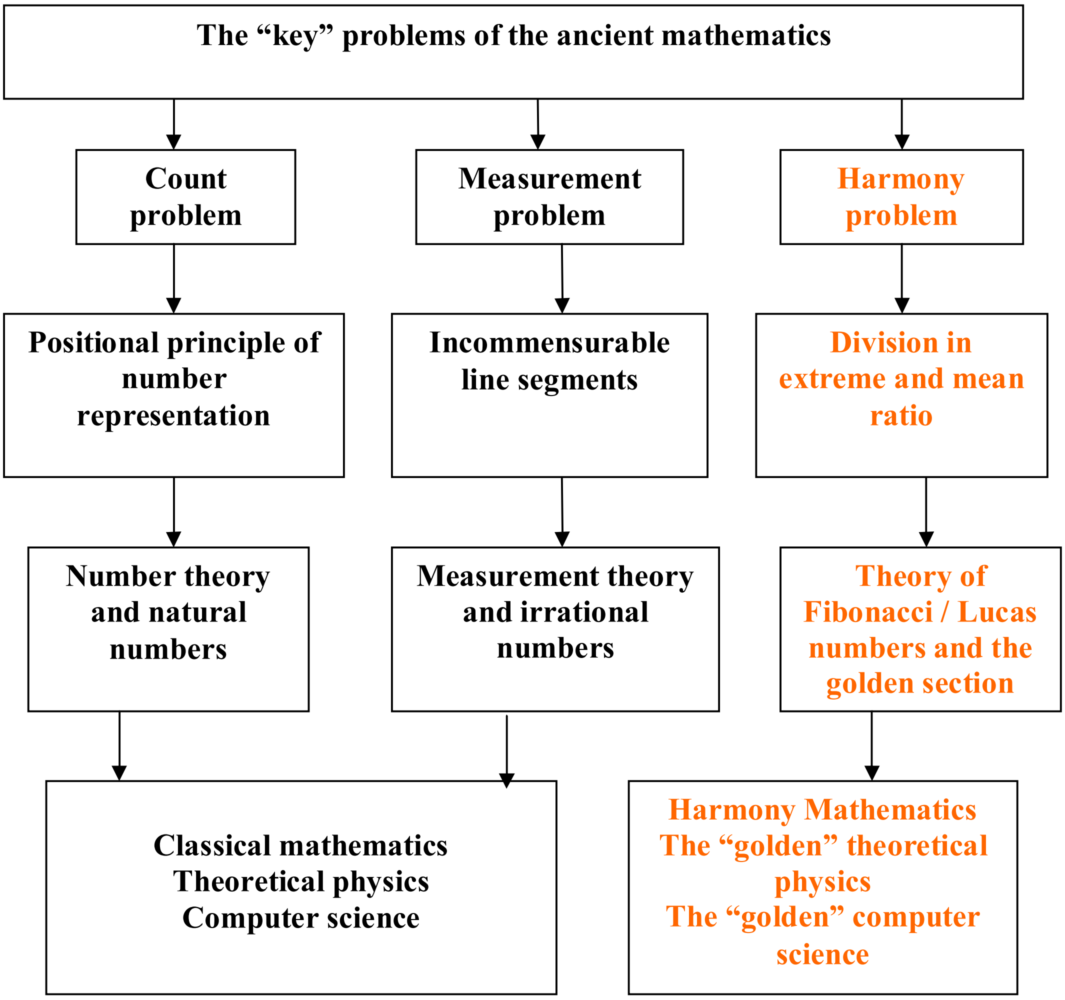
The **"strategic mistake" of 20th century mathematicians is that they took no notice of Bergman's mathematical discovery, which can be considered as the major mathematical discovery in the field of number systems (following the Babylonian discovery of the positional principle of number representation and also decimal and binary systems).**

3. Three "key" problems of mathematics and a new approach to the mathematics origins

The main purpose of the "Harmony Mathematics," which is developing by the author in recent years [4-38], is to overcome the "strategic mistakes," which arose in mathematics in process of its development. A new approach to the history of mathematics is developed in [35, 36] (see figure below).

We can see that three "key" problems – **the "count problem," the "measurement problem,"** and **the "harmony problem"** - underlie the origin of mathematics. The first two "key" problems resulted in the creation of two fundamental notions of mathematics – **natural number** and **irrational number** that underlie **classical mathematics.** The **harmony problem** connected with the **division in extreme and mean ratio** (Theorem II.11 of Euclid's *Elements*) resulted in the origin of **Harmony Mathematics** – a new interdisciplinary direction of contemporary science, which is related to contemporary mathematics, theoretical physics, and computer science. This approach leads to a conclusion, which is startling for many mathematicians. It proves to be, in parallel with classical mathematics, one more mathematical direction – **"Harmony Mathematics"** – already developing in ancient science. Similarly to "Classical Mathematics," "Harmony Mathematics" has its origin in Euclid's *Elements.* However, "Classical Mathematics" focuses its attention on the "axiomatic approach," while "Harmony Mathematics" is based on the golden section (Theorem II.11) and Platonic Solids described in Book XIII of Euclid's *Elements.* Thus, Euclid's *Elements* is the source of two independent directions in the development of mathematics – "Classical Mathematics" and "Harmony Mathematics."

For many centuries, the main focus of mathematicians was directed towards the creation of the "Classical Mathematics," which became the *Czarina of Natural Sciences.* However, the forces of many prominent mathematicians - since Pythagoras, Plato and Euclid, Pacioli, Kepler up to Lucas, Binet, Vorobyov, Hoggatt and so forth - were directed towards the development of the basic concepts and applications of Harmony Mathematics. Unfortunately, these important mathematical directions developed separately from one other. The time has come to unite "Classical Mathematics" and "Harmony Mathematics." This unusual union can lead to new scientific discoveries in mathematics and the natural sciences. Some of the latest discoveries in the natural sciences, in particular, Shechtman's quasi-crystals based on Plato's icosahedron and fullerenes (Nobel Prize of 1996) based on the Archimedean truncated icosahedron do demand this union. All mathematical theories should be united for one unique purpose: to discover and explain Nature's Laws.



A new approach to the mathematics history (see Figure above) is very important for school mathematical education. This approach introduces in a very natural manner the idea of harmony and the golden section into school mathematical education. This provides pupils access to ancient science and to its main achievement – the harmony idea – and to tell them about the most important architectural and sculptural works of ancient art based upon the golden section [including pyramid of Khufu (Cheops), Nefertiti, Parthenon, Doryphorus, Venus].

4. The generalized Fibonacci numbers and the generalized golden proportions

4.1. ***The generalized Fibonacci p-numbers, the generalized p-proportions, the generalized Binet formulas and the generalized Lucas p-numbers.*** Pascal's triangle is recognized as one of the most beautiful objects of mathematics. And we can expect further beautiful mathematical objects stemming from Pascal's triangle. In the recent decades, many mathematicians found a connection between Pascal's triangle and Fibonacci numbers independent of each other. The generalized *Fibonacci p-numbers,* which can be obtained from Pascal's triangle as its "diagonal sums" [4] are the most important of them. For a given integer p=0, 1, 2, 3, ... , they are given by the recursive relation:

$$F_p(n) = F_p(n-1) + F_p(n-p-1);\ F_p(0)=0,\ F_p(1)= F_p(2)=...= F_p(p)=1. \tag{14}$$

It is easy to see for the case p=1 that the above recursive formula is reduced to the recursive formula for classical Fibonacci numbers:

$$F_1(n) = F_1(n-1) + F_1(n-2); F_1(0)=0, F_1(1)=1. \quad (15)$$

It follows from (14) that the Fibonacci p-numbers express more complicated "harmonies" than the classical Fibonacci numbers given by (15). Note that the recursive formula (14) generates an infinite number of different recursive numerical sequences because every p generates its own recursive sequences, in particular, the binary numbers 1, 2, 4, 8, 16, … for the case p=0 and the classical Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, … for the case p=1.

It is important to note that the recursive relation (14) expresses some deep mathematical properties of Pascal's triangle (the "diagonal sums" of Pascal's triangle). The Fibonacci p-numbers are represented by the binomial coefficients as follows [4]:

$$F_p(n+1) = C_n^0 + C_{n-p}^1 + C_{n-2p}^3 + C_{n-4p}^4 + \ldots + C_{n-kp}^k + \ldots, \quad (16)$$

where the binomial coefficient $C_{n-kp}^k = 0$ for the case $k>n-kp$.

Note that for the case p=0 the formula (16) is reduced to the well-known formula of combinatorial analysis:

$$2^n = C_n^0 + C_n^1 + \ldots + C_n^n. \quad (17)$$

It is easy to prove [4] that in the limit ($n\to\infty$) the ratio of the adjacent Fibonacci p-numbers $F_p(n)/F_p(n-1)$ aims for some numerical constant, that is,

$$\lim_{n\to\infty}\frac{F_p(n)}{F_p(n-1)} = \tau_p, \quad (18)$$

where τ_p is the positive root of the following algebraic equation:

$$x^{p+1} = x^p + 1, \quad (19)$$

which for p=1 is reduced to the "golden" algebraic equation (10) given by the classical golden mean (9).

Note that the result (16) is a generalization of Kepler's formula (9) for classical Fibonacci numbers (p=1).

The positive roots of Eq.(17) were named the *golden p-proportions* [4]. It is easy to prove [4] that the powers of the golden p-proportions are connected between themselves by the following identity:

$$\tau_p^n = \tau_p^{n-1} + \tau_p^{n-p-1} = \tau_p \times \tau_p^{n-1}, \quad (20)$$

that is, each power of the golden p-proportion is connected with the preceding powers by the "additive" relation $\tau_p^n = \tau_p^{n-1} + \tau_p^{n-p-1}$ and by the "multiplicative" relation $\tau_p^n = \tau_p \times \tau_p^{n-1}$ (similar to the classical golden mean).

It is proved in [23] that the Fibonacci p-numbers can be represented in the following analytical form:

$$F_p(n) = k_1(x_1)^n + k_2(x_2)^n + \ldots + k_{p+1}(x_{p+1})^n, \quad (21)$$

where $n=0,\pm1,\pm2,\pm3,\ldots$, x_1, x_2, …, x_{p+1} are the roots of Eq. (19), and k_1, k_2, …, k_{p+1} are constant coefficients that depend on the initial elements of the Fibonacci p-series, and are solutions to the following system of algebraic equations:

$$\begin{aligned} &F_p(0) = k_1 + k_2 + \ldots + k_{p+1} = 0 \\ &F_p(1) = k_1x_1 + k_2x_2 + \ldots + k_{p+1} = 1 \\ &F_p(2) = k_1(x_1)^2 + k_2(x_2)^2 + \ldots + k_{p+1}(x_{p+1})^2 = 1 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &F_p(p) = k_1(x_1)^p + k_2(x_2)^p + \ldots + k_{p+1}(x_{p+1})^p = 1. \end{aligned} \quad (22)$$

Note that for the case p=1, the formula (21) is reduced to the Binet formula (11) for the classical Fibonacci numbers.

In [23] the generalizid Lucas p-numbers are introduced. They are repreented in the following analytical form:

$$L_p(n)=(x_1)^n+(x_2)^n+...+(x_{p+1})^n. \qquad (22)$$

where $n=0,\pm1,\pm2,\pm3,...,$ $x_1, x_2, \dots, x_{p+1}$ are the roots of Eq. (19).

Note that for the case p=1, the formula (22) is reduced to the Binet formula (12) for the classical Lucas numbers.

Directly from (22) we can deduce the following recursive relation

$$L_p(n)= L_p(n-1)+L_p(n-p-1), \qquad (23)$$

which at the seeds

$$L_p(0)=p+1 \text{ and } L_p(1)=L_p(2)=\ldots=L_p(p)=1 \qquad (24)$$

produces a new class of numerical sequences – *Lucas p-numbers.* They are a generalization of the classical Lucas numbers for the case p=1.

Thus, a study of Pascal's triangle produces the following beautiful mathematical results:

1. The generalized Fibonacci p-numbers are expressed through binomial coefficients by the graceful formula (16).
2. A new class of mathematical constants τ_p (p=0, 1, 2, 3, …), express some important mathematical properties of Pascal's triangle and possess unique mathematical properties (20).
3. A new class of algebraic equations (19), which are a wide generalization of the classical "golden" equation (10).
4. A generalization of Binet formulas for Fibonacci and Lucas p-numbers.

Discussing applications of Fibonacci p-numbers and golden p-proportions to contemporary theoretical natural sciences, we find two important applications:

1. **Asymmetric division of biological cells [45].** The authors of [45] proved that the generalized Fibonacci p-numbers can model the growth of biological cells. They conclude that *"binary cell division is regularly asymmetric in most species. Growth by asymmetric binary division may be represented by the generalized Fibonacci equation …. Our models, for the first time at the single cell level, provide a rational basis for the occurrence of Fibonacci and other recursive phyllotaxis and patterning in biology, founded on the occurrence of the regular asymmetry of binary division."*

2. **Structural harmony of systems [46].** Studying the process of system self-organization in different aspects of nature, Belarusian philosopher Eduard Soroko formulated the "Law of Structural Harmony of Systems" based on the golden p-proportions: *"The generalized golden proportions are invariants that allow natural systems in the process of their self-organization to find a harmonious structure, a stationary regime for their existence, and structural and functional stability."*

4.2. ***The generalized Fibonacci m-numbers, "Metallic Means" by Vera Spinadel, Gazale formulas and a general theory of hyperbolic functions.*** Another generalization of Fibonacci numbers was introduced recently by Vera W. Spinadel [47], Midchat Gazale [48], Jay Kappraff [49] and other scientists. We are talking about the generalized Fibonacci m-numbers that for a given positive real number m>0 are given by the recursive relation:

$$F_m(n) = mF_m(n-1) + F_m(n-2);\ F_m(0)=0,\ F_m(1)=1. \qquad (25)$$

First of all, we note that the recursive relation (25) is reduced to the recursive relation (4) for the case m=1. For other values of m, the recursive relation (25) generates an infinite number of new recursive numerical sequences.

The following characteristic algebraic equation follows from (25):

$$x^2 – mx – 1 = 0, \qquad (26)$$

which for the case $m=1$ is reduced to (10). A positive root of Eq. (26) generates an infinite number of new "harmonic" proportions – "Metallic Means" by Vera Spinadel [47], which are expressed by the following general formula:

$$\Phi_m = \frac{\sqrt{4+m^2}+m}{2}. \tag{27}$$

Note that for the case $m=1$ the formula (27) gives the classical golden mean $\Phi_1 = \frac{1+\sqrt{5}}{2}$. The metallic means possess the following unique mathematical properties:

$$\Phi_m = \sqrt{1+m\sqrt{1+m\sqrt{1+m\sqrt{\ldots}}}} \qquad \Phi_m = m + \cfrac{1}{m+\cfrac{1}{m+\cfrac{1}{m+\ldots}}}, \tag{28}$$

which are generalizations of similar properties for the classical golden mean $\Phi_1=\Phi$ ($m=1$):

$$\Phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\ldots}}}} \qquad \Phi = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}} \tag{29}$$

Note that the expressions (27), (28) and (29), without doubt, satisfy Dirac's "Principle of Mathematical Beauty" and emphasize a fundamental characteristic of both the classical golden mean and the metallic means.

Recently, by studying the recursive relation (25), the Egyptian mathematician Midchat Gazale [48] deduced the following remarkable formula given by Fibonacci m-numbers:

$$F_m(n) = \frac{\Phi_m^n - (-1)^n \Phi_m^{-n}}{\sqrt{4+m^2}}, \tag{30}$$

where $m>0$ is a given positive real number, Φ_m is the metallic mean given by (27), $n = 0, \pm1, \pm2, \pm3, \ldots$. The author of the present article named the formula (30) in [34] *formula Gazale for the Fibonacci m-numbers* after Midchat Gazale. The similar Gazale formula for the Lucas m-numbers is deduced in [34]:

$$L_m(n) = \Phi_m^n + (-1)^n \Phi_m^{-n} \tag{31}$$

First of all, we note that "Gazale formulas" (30) and (31) are a wide generalization of Binet formulas (11) and (12) for the classical Fibonacci and Lucas numbers ($m=1$).

The most important result is that the Gazale formulas (30) and (31) result in a general theory of hyperbolic functions [34].

Hyperbolic Fibonacci m-sine

$$sF_m(x) = \frac{\Phi_m^x - \Phi_m^{-x}}{\sqrt{4+m^2}} = \frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x - \left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-x}\right] \tag{32}$$

Hyperbolic Fibonacci m-cosine

$$cF_m(x) = \frac{\Phi_m^x + \Phi_m^{-x}}{\sqrt{4+m^2}} = \frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x + \left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-x}\right] \tag{33}$$

Hyperbolic Lucas m-sine

$$sL_m(x)=\Phi_m^x-\Phi_m^{-x}=\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x-\left(\frac{m+\sqrt{4+\sqrt{4+m^2}}}{2}\right)^{-x} \quad (34)$$

<u>Hyperbolic Lucas *m*-cosine</u>

$$cL_m(x)=\Phi_m^x+\Phi_m^{-x}=\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x+\left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-x} \quad (35)$$

Note that the hyperbolic Fibonacci and Lucas *m*-functions coincide with the Fibonacci and Lucas *m*-numbers for the discrete values of the variable $x=n=0,\pm1,\pm2,\pm3,\ldots$, that is,

$$F_m(n)=\begin{cases} sF_m(n) & \text{for } n=2k \\ cF_m(n) & \text{for } n=2k+1 \end{cases}$$
$$L_m(n)=\begin{cases} cL_m(n) & \text{for } n=2k \\ sL_m(n) & \text{for } n=2k+1 \end{cases} . \quad (36)$$

The formulas (32)-(35) provide an infinite number of hyperbolic models of nature because every real number *m* originates its own class of hyperbolic functions of the kind (32)-(35). As is proved in [34], these functions have, on the one hand, the "hyperbolic" properties similar to the properties of classical hyperbolic functions, and on the other hand, "recursive" properties similar to the properties of the Fibonacci and Lucas *m*-numbers (30) and (31). In particular, the classical hyperbolic functions are a partial case of the hyperbolic Lucas *m*-functions (34) and (35). For the case $m_e=e-\frac{1}{e}\approx 2.35040238\ldots$, the classical hyperbolic functions are connected with hyperbolic Lucas *m*-functions by the following simple relations:

$$sh(x)=\frac{sL_m(x)}{2} \text{ and } ch(x)=\frac{cL_m(x)}{2}. \quad (37)$$

Note that for the case $m=1$, the hyperbolic Fibonacci and Lucas *m*-functions (32)-(35) coincide with the **symmetric hyperbolic Fibonacci and Lucas functions** introduced by Alexey Stakhov and Boris Rozin in the article [20]:

<u>Symmetrical hyperbolic Fibonacci sine and cosine</u>

$$sFs(x)=\frac{\Phi^x-\Phi^{-x}}{\sqrt{5}}; \quad cFs(x)=\frac{\Phi^x+\Phi^{-x}}{\sqrt{5}} \quad (38)$$

<u>Symmetrical hyperbolic Fibonacci sine and cosine</u>

$$sLs(x)=\Phi^x-\Phi^{-x}; \quad cLs(x)=\Phi^x+\Phi^{-x} \quad (39)$$

where $\Phi=\frac{1+\sqrt{5}}{2}$.

In the book [42], the Ukrainian researcher Oleg Bodnar used Stakhov and Rozin's symmetric hyperbolic Fibonacci and Lucas functions (38) and (39) for the creation of a graceful geometric theory of phyllotaxis. This means that **the symmetrical hyperbolic Fibonacci and Lucas functions (38) and (39) and their generalization – the hyperbolic Fibonacci and Lucas *m*-functions (32)-(35) – can be ascribed to the fundamental mathematical results of modern science because they "reflect**

phenomena of Nature," in particular, phyllotaxis phenomena [42]. These functions set a general theory of hyperbolic functions that is of fundamental importance for contemporary mathematics and theoretical physics.

We propose that hyperbolic Fibonacci and Lucas *m*-functions, which correspond to the different values of *m,* can model different physical phenomena. For example, in the case of $m=2$ the recursive relation (25) is reduced to the recursive relation

$$F_2(n) = 2F_2(n-1) + F_2(n-2);\ F_2(0)=0,\ F_2(1)=1, \tag{40}$$

which gives the so-called **Pell numbers**: 0, 1, 2, 5, 12, 29, ... In this connection, the formulas for the golden *m*-proportion and hyperbolic Fibonacci and Lucas *m* -numbers take for the case $m=2$ the following forms, respectively:

$$\Phi_2 = 1+\sqrt{2} \tag{41}$$

$$sF_2(x) = \frac{\Phi_2^x - \Phi_2^{-x}}{\sqrt{8}} = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x - \left(1+\sqrt{2}\right)^{-x}\right] \tag{42}$$

$$cF_2(x) = \frac{\Phi_2^x + \Phi_2^{-x}}{\sqrt{8}} = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x + \left(1+\sqrt{2}\right)^{-x}\right] \tag{43}$$

$$sL_2(x) = \Phi_2^x - \Phi_2^{-x} = \left(1+\sqrt{2}\right)^x - \left(1+\sqrt{2}\right)^{-x} \tag{44}$$

$$cL_2(x) = \Phi_2^x + \Phi_2^{-x} = \left(1+\sqrt{2}\right)^x + \left(1+\sqrt{2}\right)^{-x}. \tag{45}$$

It is appropriate to give the following comparative table, which gives a relationship between the golden mean and metallic means as new mathematical constants of Nature.

The Golden Mean (*m*=1)	**The Metallic Means (*m*>0)**
$\Phi = \frac{1+\sqrt{5}}{2}$	$\Phi_m = \frac{\sqrt{4+m^2}+m}{2}$
$\Phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\ldots}}}}$	$\Phi_m = \sqrt{1+m\sqrt{1+m\sqrt{1+m\sqrt{\ldots}}}}$
$\Phi = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}$	$\Phi_m = m+\cfrac{1}{m+\cfrac{1}{m+\cfrac{1}{m+\ldots}}}$
$\Phi^n = \Phi^{n-1}+\Phi^{n-2} = \Phi\times\Phi^{n-1}$	$\Phi_m^n = m\Phi_m^{n-1}+\Phi^{n-2} = \Phi_m\times\Phi_m^{n-1}$
$F_n = \frac{\Phi^n-(-1)^n\Phi^{-n}}{\sqrt{5}}$	$F_m(n) = \frac{\Phi_m^n-(-1)^n\Phi_m^{-n}}{\sqrt{4+m^2}}$
$L_n = \Phi^n+(-1)^n\Phi^{-n}$	$L_m(n) = \Phi_m^n+(-1)^n\Phi_m^{-n}$
$sFs(x) = \frac{\Phi^x-\Phi^{-x}}{\sqrt{5}};\quad cFs(x) = \frac{\Phi^x+\Phi^{-x}}{\sqrt{5}}$	$sF_m(x) = \frac{\Phi_m^x-\Phi_m^{-x}}{\sqrt{4+m^2}};\ cF_m(x) = \frac{\Phi_m^x+\Phi_m^{-x}}{\sqrt{4+m^2}}$
$sLs(x) = \Phi^x-\Phi^{-x};\ cLs(x) = \Phi^x+\Phi^{-x}$	$sL_m(x) = \Phi_m^x-\Phi_m^{-x};\ cL_m(x) = \Phi_m^x+\Phi_m^{-x}$

A beauty of these formulas is charming. This gives a right to suppose that Dirac's "Principle of Mathematical Beauty" can be applicable fully to the metallic means and hyperbolic Fibonacci and Lucas *m*-functions. And this, in its turn, gives hope that these mathematical results can become a base for theoretical natural sciences.

A general theory of hyperbolic functions given by (32)-(35) can lead to the following scientific theories of a fundamental character: **(1) Lobachevski's "golden" geometry; (2) Minkovski's**

"golden" geometry as an original interpretation of Einstein's special theory of relativity. In Lobachevski's "golden" geometry and Minkovski's "golden" geometry, the processes of the real world are modeled, in the general case, by the hyperbolic Fibonacci and Lucas *m*-functions (32)-(35). The classical Lobachevski geometry, Minkowski geometry and Bodnar geometry [42] are partial cases of this general hyperbolic geometry. We propose that this approach is of great importance for contemporary mathematics and theoretical physics and could become the source of new scientific discoveries.

It is clear that all mathematical results given by formulas (25)-(45) satisfy Dirac's "Principle of Mathematical Beauty." There can be no doubt that these beautiful mathematical results will be widely used in modern theoretical natural sciences. Bodnar's geometry [42] provides hope of this.

5. A new geometric definition of number

5.1. ***Euclidean and Newtonian definition of a real number.*** The first definition of a number was made in Greek mathematics. We are talking about the "Euclidean definition of natural number":

$$N = \underbrace{1+1+...+1}_{N} \qquad (46)$$

In spite of the utmost simplicity of the *Euclidean definition* (46), it played a decisive role in mathematics, in particular, in number theory. This definition underlies many important mathematical concepts, for example, the concept of *prime* and *composite* numbers, and also *divisibility* that is one of the major concepts of number theory. Over the centuries, mathematicians developed and defined more exactly the concept of a number. In the 17th century, that is, in the period of the creation of new science, in particular, new mathematics, a number of methods for the study of "continuous" processes were developed and the concept of a real number again moves into the foreground. Most clearly, a new definition of this concept was given by Isaac Newton, one of the founders of mathematical analysis, in his *Arithmetica Universalis* (1707):

"We understand a number not as a set of units, but as the abstract ratio of one magnitude to another magnitude of the same kind taken for the unit."

This formulation gives us a general definition of numbers, rational and irrational. For example, the binary system

$$N = a_n 2^{n-1} + a_{n-1} 2^{n-2} + ... + a_i 2^{i-1} + ... + a_1 2^0 \qquad (47)$$

is an example of *Newton's definition,* when we choose the number of 2 for the unit and represent a number as the sum of the powers of number 2.

5.2. ***Number systems with irrational radices as a new definition of real number.*** Let us consider the set of the powers of the golden *p*-proportions:

$$S = \{\tau_p^i, p=0, 1, 2, 3, ...; i=0, \pm1, \pm2, \pm3, ...\}. \qquad (48)$$

By using (48), we can construct the following method of positional representation of real number A:

$$A = \sum_i a_i \tau_p^i \qquad (49)$$

where a_i is the binary numeral of the *i*-th digit; τ_p^i is the weight of the *i*-th digit; τ_p is the radix of the numeral system (47), $i = 0, \pm1, \pm2, \pm3, \dots$. The positional representation (49) is called *code of the golden p-proportion* [6, 17].

Note that for the case p=0 the sum (49) is reduced to the classical binary representation of real numbers:

$$A=\sum_i a_i 2^i \tag{50}$$

For the case p=1, the sum (49) is reduced to Bergman's system (13). For the case $p\to\infty$, the sum (49) strives for expression similar to (46).

In the author's article [17], a new approach to geometric definition of real numbers based on (49) was developed. A new theory of real numbers based on the definition (49) contains a number of unexpected results concerning number theory. Let us study these results as applied to Bergman's system (13). We shall represent a natural number N in Bergman's system (13):

$$N=\sum_i a_i \tau^i\,. \tag{51}$$

The following theorems are proved in [17]:

1. Every natural number N can be represented in the form (51) as a finite sum of the golden powers τ^i (i=0,±1,±2,±3,…). Note that this theorem is not a trivial property of natural numbers.

2. ***Z*-property of natural numbers.** If we substitute in (51) the Fibonacci number F_i for the power of the golden mean τ^i {i=0,±1,±2,±3,…}, then the sum that appears as a result of such a substitution is equal to 0 independent of the initial natural number N, that is,

$$\sum_i a_i F_i = 0. \tag{52}$$

3. ***D*-property of natural numbers.** If we substitute in (51) the Lucas number L_i for the power of the golden mean τ^i {i=0,±1,±2,±3,…}, then the sum that appears as a result of such a substitution is equal to the double sum (51) independent of the initial natural number N, that is,

$$\sum_i a_i L_i = 2N. \tag{53}$$

4. ***F*-code of natural number *N*.** If we substitute in (51) the Fibonacci number F_{i+1} for the power of the golden mean τ^i {i=0,±1,±2,±3,…}, then the sum that appears as a result of such a substitution is a new positional representation of the same natural number N called the *F-code of natural number N,* that is,

$$N=\sum_i a_i F_{i+1}\left(i=0,\pm1,\pm2,\pm3,\ldots\right). \tag{54}$$

5. ***L*-code of natural number *N*.** If we substitute in (51) the Lucas number L_{i+1} for the power of the golden mean τ^i {i=0,±1,±2,±3,…}, then the sum that appear as a result of such substitution is a new positional representation of the same natural number N called *L-code of natural number N,* that is,

$$N=\sum_i a_i L_{i+1}\left(i=0,\pm1,\pm2,\pm3,\ldots\right). \tag{55}$$

Note that similar properties are proved for the code of the golden p-proportion given by (49).

Thus, we have discovered new properties of natural numbers (***Z*-property, *D*-property, *F*- and *L*-codes)** that confirm the fruitfulness of such an approach to number theory [17]. These results are of great importance for computer science and could become a source for new computer projects.

As the study of the positional binary and decimal systems are an important part of mathematical education, the number systems with irrational radices given by (13) and (49) are of general interest for mathematical education.

6. Fibonacci and "golden" matrices

6.1. ***Fibonacci matrices.*** For the first time, a theory of the *Fibonacci Q-matrix* was developed in the book [50] written by the eminent American mathematician **Verner Hoggatt** – founder of the *Fibonacci Association* and *The Fibonacci Quarterly*.

The article [51] devoted to the memory of Verner E. Hoggatt contained a history and extensive bibliography of the Q-matrix and emphasized Hoggatt's contribution to its development. Although the name of the Q-matrix was introduced before Verner E. Hoggatt, he was the first mathematician who appreciated the mathematical beauty of the Q-matrix and introduced it into Fibonacci number theory. Thanks to Hoggatt's work, the idea of the Q-matrix *"caught on like wildfire among Fibonacci enthusiasts. Numerous papers appeared in 'The Fibonacci Quarterly' authored by Hoggatt and/or his students and other collaborators where the Q-matrix method became the central tool in the analysis of Fibonacci properties"* [51].

The Q-matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad (56)$$

is a generating matrix for Fibonacci numbers and the following wonderful properties:

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n+1} \end{pmatrix} \quad (57)$$

$$Det\, Q^n = F_{n+1}F_{n+1} - F_n^2 = (-1)^n \quad (58)$$

Note that there is a direct relation between the Cassini formula (8) and the formula (58) given for the determinant of the matrix (57).

In article [15], the author introduced a generating matrix for Fibonacci p-numbers called Q_p-*matrix* (p=0, 1, 2, 3, …):

$$Q_p = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad (59)$$

The following properties of Fibonacci p-numbers are proved in [15]:

$$Q_p^n = \begin{pmatrix} F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix} \quad (60)$$

$$Det\, Q_p^n = (-1)^{pn}, \quad (61)$$

where p=0, 1, 2, 3, ... ; and n=0, ±1, ±2, ±3, ...

The generating matrix G_m for the Fibonacci m-numbers $F_m(n)$

$$G_m = \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} \quad (62)$$

was introduced in [33]. The following properties of the G_m-matrix (62) are proved in [33]:

$$G_m^n = \begin{pmatrix} F_m(n+1) & F_m(n) \\ F_m(n) & F_m(n-1) \end{pmatrix} \tag{63}$$

$$Det\, G_m^n = (-1)^n . \tag{64}$$

The general property of the Fibonacci Q-, Q_p-, and G_m-matrices consists of the following. The determinants of the Fibonacci Q-, Q_p-, and G_m-matrices and all their powers are equal to +1 or -1. This unique property emphasizes mathematical beauty in the Fibonacci matrices and unites them into a special class of matrices, which are of fundamental interest for matrix theory.

6.2. ***The "golden" matrices.*** Integer numbers – the classical Fibonacci numbers, the Fibonacci p- and m-numbers - are elements of Fibonacci matrices (57), (60) and (63). In [28] a special class of the square matrices called **"golden" matrices** was introduced. Their peculiarity is the fact that the hyperbolic Fibonacci functions (38) or the hyperbolic Fibonacci m-functions (32) and (33) are elements of these matrices. Let us consider the simplest of them [28]:

$$Q^{2x} = \begin{pmatrix} cFs(2x+1) & sFs(2x) \\ sFs(2x) & cFs(2x-1) \end{pmatrix}; \qquad Q^{2x+1} = \begin{pmatrix} sFs(2x+2) & cFs(2x+1) \\ cFs(2x+1) & sFs(2x) \end{pmatrix}. \tag{65}$$

If we calculate the determinants of the matrices (65), we obtain the following unusual identities:

$$\begin{aligned} Det\, Q^{2x} &= cFs(2x+1)\times cF(2x-1) - [sFs(2x)]^2 = 1 \\ Det\, Q^{2x+1} &= sFs(2x+2)\times sF(2x) - [cFs(2x+1)]^2 = -1 \end{aligned}. \tag{66}$$

The "golden" matrices based on hyperbolic Fibonacci m-functions (32) and (33) take the following form [33]:

$$G_m^{2x} = \begin{pmatrix} cF_m(2x+1) & sF_m(2x) \\ sF_m(2x) & cF_m(2x-1) \end{pmatrix}; \qquad G_m^{2x+1} = \begin{pmatrix} sF_m(2x+2) & cF_m(2x+1) \\ cF_m(2x+1) & sF_m(2x) \end{pmatrix}. \tag{67}$$

It is proved [33] that the "golden" G_m-matrices (67) possess the following unusual properties:

$$Det\, G_m^{2x} = 1; \quad Det\, G_m^{2x+1} = -1 . \tag{68}$$

The mathematical beauty of "golden" matrices (65) and (67) are confirmed by their unique mathematical properties (66) and (68).

7. Applications in computer science

7.1. ***Fibonacci codes, Fibonacci arithmetic and Fibonacci computers.*** The concept of *Fibonacci computers* suggested by the author in a speech "Algorithmic Measurement Theory and Foundations of Computer Arithmetic" given to the joint meeting of Computer and Cybernetics Societies of Austria (Vienna, March 1976) and described in the book [4] is one of the more important ideas of modern computer science. The essence of the concept amounts to the following: modern computers are based on a binary system (50), which represents all numbers as sums of the binary numbers with binary coefficients, 0 and 1. However, the binary system (50) is non-redundant and does not allow for detection of errors, which could appear in the computer during the process of its exploitation. In order to eliminate this shortcoming, the author suggested [4] the use of *Fibonacci p-codes*

$$N = a_nF_p(n) + a_{n-1}F_p(n-1) + ... + a_iF_p(i) + ... + a_1F_p(1), \tag{69}$$

where N is a natural number, $a_i \in \{0, 1\}$ is a binary numeral of the i-th digit of the code (69); n is the digit number of the code (69); $F_p(i)$ is the i-th digit weight calculated in accordance with the recursive relation (14).

Thus, Fibonacci p-codes (68) represent all numbers as the sums of Fibonacci p-numbers with binary coefficients, 0 and 1. In contrast to the binary number system (47), the Fibonacci p-codes (69) are redundant positional methods of number representation. This redundancy can be used for checking different transformations of numerical information in the computer, including arithmetical operations. A Fibonacci computer project was developed by the author in the former Soviet Union from 1976 right up to the disintegration of the Soviet Union in 1991. Sixty-five foreign patents in the U.S., Japan, England, France, Germany, Canada and other countries are official juridical documents, which confirm Soviet priority in Fibonacci computers.

7.2. ***Ternary mirror-symmetrical arithmetic.*** Computers can be constructed by using different number systems. The ternary computer "Setun" designed in Moscow University in 1958 was the first computer based not on a binary system but on a ternary system [52]. The ternary mirror-symmetrical number system [16] is an original synthesis of the classical ternary system [52] and Bergman's system (13) [44]. It represents integers as the sum of golden mean squares with ternary coefficients {-1, 0, 1}. Each ternary representation consists of two parts that are disposed symmetrically with respect to the 0th digit. However, one part is mirror-symmetrical to another part. At the increase of a number, its ternary mirror-symmetrical representation is expanding symmetrically to the left and to the right with respect to 0-th digit. This unique mathematical property produces a very simple method for checking numerical information in computers. It is proved that the mirror-symmetric property is invariant with respect to arithmetical operations, that is, the results of all arithmetical operations have mirror-symmetrical form. This means that the mirror-symmetrical arithmetic can be used for designing self-controlling and fault-tolerant processors and computers.

The article *Brousentsov's Ternary Principle, Bergman's Number System and Ternary Mirror-Symmetrical Arithmetic* [16] published in "The Computer Journal" (England) got a high approval from two outstanding computer specialists - **Donald Knut,** Professor-Emeritus of Stanford University and the author of the famous book *The Art of Computer Programming*, and **Nikolay Brousentsov,** Professor at Moscow University, a principal designer of the fist ternary computer "Setun." And this fact gives hope that the ternary mirror-symmetrical arithmetic [16] can become a source of new computer projects in the near future.

7.3. ***A new theory of error-correcting codes based upon Fibonacci matrices.*** The error-correcting codes [53, 54] are used widely in modern computer and communication systems for the protection of information from noise. The main idea of error-correcting codes consists in the following [53, 54]. Let us consider the initial code combination that consists of n data bits. We add to the initial code combination m error-correction bits and build up the k-digit code combination of the error-correcting code, or (k,n)-code, where $k = n+m$. The error-correction bits are formed from the data bits as the sums by module 2 of certain groups of the data bits. There are two important coefficients, which characterize an ability of error-correcting codes to detect and correct errors [53].

<u>The potential detecting ability</u>

$$S_d = 1 - \frac{1}{2^m} \tag{70}$$

<u>The potential correcting ability</u>

$$S_c = \frac{1}{2^n}, \tag{71}$$

where m is the number of error-correction bits, n is the number of data bits.

The formula (71) shows that the coefficient of potential correcting ability diminishes potentially to 0 as the number n of data bits increases. For example, the Hamming (15,11)-code allows one to to detect $2^{11}\times(2^{15} - 2^{11}) = 62{,}914{,}560$ erroneous transitions; at that rate it can only correct $2^{15} - 2^{11} = 30{,}720$ erroneous transitions, that is, it can correct only 30,720/62,914,560 = 0.0004882 (0.04882%) erroneous transitions. If we take n=20, then according to (71) the potential correcting ability of the error-correcting (k,n)-code diminishes to 0.00009%. Thus, the potential correcting ability of the classical error-correcting codes [53, 54] is very low. This conclusion is of fundamental importance! One more fundamental shortcoming of all known error-correcting codes is the fact that the very small information elements, bits and their combinations are objects of detection and correction.

The new theory of error-correcting codes [6, 26] that is based on Fibonacci matrices has the following advantages in comparison to the existing theory of algebraic error-correcting codes [53, 54]:

1. The Fibonacci coding/decoding method is reduced to matrix multiplication, that is, to the well-known algebraic operation that is carried out so well in modern computers.
2. The main practical peculiarity of the Fibonacci encoding/decoding method is the fact that large information units, in particular, matrix elements, are objects of detection and correction of errors.
3. The simplest Fibonacci coding/decoding method (p=1) can guarantee the restoration of all "erroneous" (2×2)-code matrices having "single," "double" and "triple" errors.
4. The potential correcting ability of the method for the simplest case p=1 is between 26.67% and 93.33% which exceeds the potential correcting ability of all known algebraic error-correcting codes by 1,000,000 or more times. This means that a new coding theory based upon the matrix approach is of great practical importance for modern computer science.

7.4. ***The "golden" cryptography.*** All existing cryptographic methods and algorithms [55] were created for "ideal conditions" when we assume that the coder, communication channel, and the decoder operate "ideally," that is, the coder carries out the "ideal" transformation of plaintext into ciphertext, the communication channel transmits "ideally" ciphertext from the sender to the receiver and the decoder carries out the "ideal" transformation of ciphertext into plaintext. It is clear that the smallest breach of the "ideal" transformation or transmission is a catastrophe for the cryptosystem. All existing cryptosystems based upon both symmetric and public-key cryptography have essential shortcomings because they do not have in their principles and algorithms an inner "checking relation" that allows checking the informational processes within the cryptosystems.

The "golden" cryptography developed in [33] is based upon the use of matrix multiplication by the special "golden" G_m-matrices (65). This method of cryptography possesses unique mathematical properties (68) that connect the determinants of the initial matrix (plaintext) and the code matrix (ciphertext). Thanks to these properties, we can check all informational processes in the cryptosystem, including encryption, decryption and transmission of the ciphertext via the channel. Such an approach can result in the designing of simple and reliable cryptosystems for technical realization. **Thus, "golden" cryptography opens with a new stage in the development of cryptography – designing super-reliable cryptosystems.**

8. Fundamental discoveries of modern science based upon the golden section and "Platonic Solids"

8.1. ***Shechtman's quasi-crystals.*** It is necessary to note that right up to the last quarter of the 20th century the use of the golden mean in theoretical science, in particular, in theoretical physics, was very rare. In order to be convinced of this, it is enough to browse 10 volumes of *Theoretical Physics* by Landau and Lifshitz. We cannot find any mention about the golden mean and Platonic solids. The situation in theoretical science changed following the discovery of *Quasi-crystals* by the Israel researcher *Dan Shechtman* in 1982 [56].

One type of quasi-crystal was based upon the regular icosahedron described in Euclid's *Elements*! Quasi-crystals are of revolutionary importance for modern theoretical science. First of all, this discovery is the moment of a great triumph for the "icosahedron-dodecahedron doctrine," which proceeds throughout all the history of the natural sciences and is a source of deep and useful scientific ideas. Secondly, the quasi-crystals shattered the conventional idea that there was an insuperable watershed between the mineral world where the "pentagonal" symmetry was prohibited, and the living world, where the "pentagonal" symmetry is one of most widespread. Note that Dan Shechtman published his first article about the quasi-crystals in 1984, that is, exactly 100 years after the publication of Felix Klein's *Lectures on the Icosahedron* ... (1884) [43]. This means that this discovery is a worthy gift to the centennial anniversary of Klein's book [43], in which Klein predicted the outstanding role of the icosahedron in the future development of science.

8.2. ***Fullerenes (Nobel Prize for chemistry of 1996).*** The discovery of *fullerenes* is one of the more outstanding scientific discoveries of modern science. This discovery was made in 1985 by *Robert F. Curl*, *Harold W. Kroto* and *Richard E. Smalley*. The title "fullerenes" refers to the carbon molecules of the type C_{60}, C_{70}, C_{76}, C_{84}, in which all atoms are on a spherical or spheroid surface. In these molecules the atoms of carbon are located at the vertexes of regular hexagons and pentagons that cover the surface of a sphere or spheroid. The molecule C_{60} plays a special role amongst fullerenes. This molecule is based upon the Archimedean truncated icosahedron. The molecule C_{60} is characterized by the greatest symmetry and as a consequence is of the greatest stability. In 1996 *Robert F. Curl*, *Harold W. Kroto* and *Richard E. Smalley* won the Nobel Prize in chemistry for this discovery.

8.3. ***El-Naschie's E-infinity theory.*** Prominent theoretical physicist and engineering scientist Mohammed S. El Naschie is a world leader in the field of golden mean applications to theoretical physics, in particular, quantum physics [57] – [64]. El Naschie's discovery of the golden mean in the famous physical two-slit experiment — which underlies quantum physics — became the source of many important discoveries in this area, in particular, of *E*-infinity theory. It is also necessary to note that the important contribution of Slavic researchers in this area. The book [65] written by Belarusian physicist Vasyl Pertrunenko is devoted to applications of the golden mean in quantum physics and astronomy.

8.4. ***Bodnar's geometry.*** According to the law of phyllotaxis the numbers on the left-hand and right-hand spirals on the surface of phyllotaxis objects are always adjacent Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Their ratios 1/1, 2/1, 3/2, 5/3, 8/5, 13/8, 21/13, ... are called a *symmetry order* of phyllotaxis objects. The phyllotaxis phenomena excited the best minds of humanity during the centuries since Johannes Kepler. The "puzzle of phyllotaxis" consists of the fact that a majority of bio-forms change their phyllotaxis orders during their growth. It is known, for example, that sunflower disks that are located on different levels of the same stalk have different phyllotaxis orders; moreover, the greater the age of the disk, the higher its phyllotaxis order. This means that during the growth of the phyllotaxis object, a natural modification (increase) in symmetry happens and this modification of symmetry obeys the law:

$$\frac{2}{1} \to \frac{3}{2} \to \frac{5}{3} \to \frac{8}{5} \to \frac{13}{8} \to \frac{21}{13} \to \dots . \tag{72}$$

The law (42) is called *dynamic symmetry*.

Recently Ukrainian researcher Oleg Bodnar developed a very interesting geometric theory of phyllotaxis [42]. He proved that phyllotaxis geometry is a special kind of non-Euclidean geometry based upon the "golden" hyperbolic functions similar to hyperbolic Fibonacci and Lucas functions (38) and (39). Such approach allows one to explain geometrically how the "Fibonacci spirals" appear on the surface of phyllotaxis objects (for example, pine cones, ananas, and cacti) in the process of their growth and thus dynamic symmetry (72) appears. Bodnar's geometry is of essential importance because it concerns fundamentals of the theoretical natural sciences, in particular, this discovery gives a strict geometrical explanation of the phyllotaxis law and dynamic symmetry based upon Fibonacci numbers.

8.5. *Petoukhov's "golden" genomatrices.* The idea of the genetic code is amazingly simple. The record of the genetic information in ribonucleic acids (RNA) of any living organism, uses the "alphabet" that consists of four "letters" or the nitrogenous bases: *Adenine* (*A*), *Cytosine* (*C*), *Guanine* (*G*), *Uracil* (*U*) (in DNA instead of the *Uracil* it uses the related *Thymine* (*T*)). Petoukhov's article [66] is devoted to the description of an important scientific discovery—the *golden genomatrices*, which affirm the deep mathematical connection between the golden mean and the genetic code.

Thus, there are enough confirmations that the golden mean and its generalizations – the golden *p-* and *m*-proportions – underlie different fields of the theoretical natural sciences (crystallography – *Shechtman's quasi-crystals*, chemistry – *fullerenes,* quantum physics – *El-Naschie's E-infinity,* botany – *Bodnar's geometry,* and genetics - *Petoukhov's "golden" genomatrices*).

9. Conclusion

The following conclusions follow from present research:

9.1. Mathematics and mathematical education begin with Euclid's *Elements*, the greatest mathematical work of Greek mathematics. It included geometric axioms, the beginnings of algebra, theory of numbers and theory of irrationals, a theory of the "golden section" (Theorem II.11), "Platonic Solids," and Book XIII which expresses the Harmony of the Cosmos based upon Plato's "Cosmology." According to the opinion of **Proclus Diadochus** (412-485), the leading commentator on Euclid's *Elements,* the creation of a theory of "Platonic Solids" was the main objective of Euclid's *Elements.* This theory was presented in finalist conclusion, that is, the 13-th book of Euclid's *Elements*. All previous Books of Euclid's *Elements* were devoted to a description of the "auxiliary material," which reflected the most important mathematical achievements in Greek mathematics. By using this "auxiliary material," Euclid solved the main task - the creation of a strictly geometrical theory of the "Platonic Solids". **Thus, Euclid's *Elements* is the first attempt to reflect in mathematics the major scientific idea of Greek science, the idea of Harmony.**

9.2. Unfortunately, the "Classical Mathematics" and the "classical mathematical education" took in only the "auxiliary material" (axioms of geometry, the beginning of algebra, theory of numbers and irrationals) from Euclid's *Elements* for further creation of mathematics and programs of mathematical education. Thus, the ideas of harmony, the "golden section" and "Platonic Solids" were excluded from mathematics and mathematical education because they underlie Pythagoras' and Plato's system of cosmic proportions, which were considered sometimes in literature "as a curious result of unrestrained and preposterous fantasy" (Alexey Losev). As a result of such an approach, the Mathematics of Harmony based on the "golden section" and Platonic Solids was excluded from mathematics and mathematical education for 2.5 millennia. **This exclusion of the ideas of harmony and the "golden section" from mathematics and mathematical education are major "strategic mistakes" in their development.**

9.3. **The ideas of Universal Harmony and the Golden Section**, which return us to the Pythagorean doctrine about numerical harmony of the Universe, is the most ancient scientific paradigm arising during this same period. This belongs to the category of "eternal" scientific problems, an interest which never vanished from the field of scientific vision, but especially increased during periods of the highest flowering of human culture. **There are all indications to believe that the last quarter of the 20th century and the beginning of the 21st century became the initial stages of an original Renaissance or rebirth of this most ancient scientific paradigm.** This tendency is confirmed by modern scientific discoveries based on the "golden section," Fibonacci and Lucas numbers, and

Platonic Solids, in particular, the dodecahedron and icosahedron (quasi-crystals, fullerenes, Bodnar's geometry of phyllotaxis, Pethoukhov's "golden" genomatrices, El Naschie's E-infinity theory, and so forth). Modern science, in which processes of differentiation prevail, requires a certain interdisciplinary direction, which can best unite different areas of science, art and technology. It is primarily the **Mathematics of Harmony** [4-38] that can fulfill this role for modern science.

9.4. Since the beginning of its origin (Euclid's *Elements*), the Mathematics of Harmony was based upon the "Principle of Mathematical Beauty," and more fully developed later by Dirac. All the initial results of the Mathematics of Harmony (the golden mean, Fibonacci and Lucas numbers and Platonic Solids) are beautiful. And all next results of the Mathematics of Harmony - the generalized Fibonacci and Lucas numbers, the golden p-proportions and metallic means, Fibonacci and "golden" matrices and so on - satisfy to Dirac's "Principle of Mathematical Beauty." This fact is a prerequisite for the widespread use of the Mathematics of Harmony in the theoretical natural sciences.

9.5. The overcoming of the main "strategic mistake" in the development of mathematics of the 20th century, namely **the overcoming of the severance of the relationship between mathematics and theoretical natural sciences,** can be considered the main result of the Mathematics of Harmony. The Mathematics of Harmony offers to theoretical natural sciences a tremendous amount of new recursive relations and new mathematical constants, which can be used by the theoretical natural sciences for the creation of new mathematical models of natural phenomena and processes. As an example we can consider the modern theory of the process of biological cell division. As shown in [45], the process of cell division is of a fundamentally asymmetric character; the cell division is based upon the Fibonacci p-numbers corresponding to the cases p=2 and p=3. However, **new hyperbolic models of Nature based upon the golden mean, Fibonacci and Lucas numbers and their generalizations - the metallic means and Fibonacci and Lucas *m*-numbers** - are the greatest confirmation of the efficiency of the application of Harmony Mathematics to theoretical natural sciences. A new class of hyperbolic functions [9, 13, 20, 33] develop and generalize the classical "Fibonacci number theory" and transforms it into a continuous theory. The new geometrical theory of phyllotaxis, developed by Ukrainian architect Oleg Bodnar [42], is a brilliant confirmation of the efficiency of hyperbolic Fibonacci and Lucas functions for the simulation of processes proceeding in wildlife. **The "golden" hyperbolic world based on hyperbolic Fibonacci and Lucas functions and Bodnar's geometry exists objectively and irrespective of our consciousness. This world with surprising persistence shows itself, first of all, in wildlife, in particular, on surfaces of pine cones, pineapples, cacti, heads of sunflowers, and baskets of flowers in the form of phyllotaxis spirals based upon Fibonacci and Lucas numbers.**

9.6. The Mathematics of Harmony can lead to new projects in the field of computers (**Fibonacci computers, ternary mirror-symmetrical computers**) and in the field of communication systems (**a new theory of error-correcting codes, the "golden" cryptography,** and so on).

9.7. The Mathematics of Harmony can influence the development of modern mathematical and general education. A wide introduction of the Harmony paradigm into mass consciousness can be carried out by the introduction of a complex of courses on the Mathematics of Harmony into different educational programs of schools, colleges and universities by differentiating them on various levels of complexity.

9.8. Since the lecture **"The Golden Section and Modern Harmony Mathematics"** [14] given by the author at the session of the 7th International conference "Fibonacci numbers and their applications" (Austria, Graz, July 1996), one question is worrying the author more and more. This question became even more topical in the process of the creation of author's book **"The Mathematics of Harmony. From Euclid to Contemporary Mathematics and Computer Science"** [11]. Soroko proved [46] that the generalized golden p-proportions are a base of the "Law of Structural Harmony of Systems." Bodnar proved that the hyperbolic Fibonacci functions based on the golden mean are "natural" functions, which exist in Nature independently of our consciousness. Petoukhov proved [66] that the golden mean is a base of genetic code. But then we can ask the question: who did create these mathematical constants and functions and who did embody them into natural objects such as biological cells, genetic code, pine cones, cacti, pineapples, and heads of sunflower, which existed in Nature long before "Homo Sapiens" origin? Also we can ask the same question concerning the "Metallic Means" by Vera Spinadel and the hyperbolic Fibonacci and Lucas *m*-functions. We have serious reason to suppose that the hyperbolic Fibonacci and Lucas *m*-functions are "natural" functions. They exist in Nature independently of our consciousness, however they are not discovered for the time being by modern science. And the main problem of science is to discover these functions in Nature as soon as possible. And we have a right to expect new fundamental discoveries by moving in this direction.

In conclusion we can offer the following

FUNDAMENTAL SUGGESTION:

If the generalized golden *p*-proportions and metallic means (including the golden mean) are new mathematical constants of Nature and the hyperbolic Fibonacci and Lucas *m*-functions are really "natural" functions, then they are cogent argument of the existence of the Universal Reason, which constructs a "Nature Building" according to the Laws of mathematical beauty? And possibly, the Polish cosmologist and Catholic priest Michael Heller, Laureate of Templeton Prize-2008, is right, when he wrote the following: *"Amongst my numerous fascinations, two have most imposed themselves and proven more time resistant than others: science and religion. I am also too ambitious. I always wanted to do the most important things, and what can be more important than science and religion. Science gives us Knowledge and religion gives us Meaning. Both are prerequisites of the decent existence. The paradox is that these two great values seem often to be in conflict. I am frequently asked how I could reconcile them with each other. When such a question is posed by a scientist or a philosopher, I invariably wonder how educated people could be so blind not to see that science does nothing else but explores God's creation"* [67].

Acknowledgements
The author would like to express acknowledgements to Professor Scott Olsen (U.S.) for the English edition of this article and valuable remarks.

References

1. Arnold V. A complexity of the finite sequences of zeros and units and geometry of the finite functional spaces. Lecture at the session of the Moscow Mathematical Society, May 13, 2007 http://elementy.ru/lib/430178/430281
2. Olsen S.A. The Golden Section: Natures Greatest Secret. New York: Walker & Company, 2006.
3. Metaphysics: Century XXI (compiler and editor Y.S. Vladimirov). Moscow: "BINOM" publishing house, 2006 (in Russian).
4. Stakhov A.P. Introduction into algorithmic measurement theory. Moscow: Soviet Radio, 1977 (in Russian)
5. Stakhov A.P. Algorithmic measurement theory. Moscow: Publisher "Znanie," series "Mathematics and Cybernetics," issue No 6, 1979 (in Russian)
6. Stakhov A.P. Codes of the golden proportion. Moscow: Radio and communications, 1984 (in Russian).
7. Stakhov A.P., Massingua V., Sluchenkova A.A. Introduction into Fibonacci Coding and Cryptography. Kharkov: Publisher "Osnova," 1999.
8. Stakhov A.P., Sluchenkova A.A. "Museum of Harmony and Golden Section: mathematical connections in Nature, Science and Art." Vinnitsa: Publisher "ITI," 2003.
9. Stakhov A.P. Hyperbolic Fibonacci and Lucas Functions: a New Mathematics for Living Nature. Vinnitsa: Publisher "ITI," 2003.
10. Stakhov A.P., Sluchenkova A.A., Scherbakov I.G. Da Vinci code and Fibonacci series. Saint Petersburg: Publisher "Piter," 2006.

11. Stakhov A.P. The Mathematics of Harmony. From Euclid to Contemporary Mathematics and Computer Science. New Jersey. London. Singapore. Hong Kong: World Scientific (in press)
12. Stakhov A.P. The Golden Section in the measurement theory. An International Journal "Computers & Mathematics with Applications," Volume 17, No 4-6, 1989.
13. Stakhov A.P., Tkachenko I.S. Hyperbolic Fibonacci trigonometry. Reports of the Ukrainian Academy of Sciences, Volume 208, No 7, 1993 (in Russian).
14. Stakhov A.P. The Golden Section and Modern Harmony Mathematics. Applications of Fibonacci Numbers, Volume 7, 1998.
15. Stakhov A.P. A generalization of the Fibonacci *Q*-matrix. Reports of the Ukrainian Academy of Sciences, №9, 1999, p. 46-49 (in Russian).
16. Stakhov A.P. Brousentsov's ternary principle, Bergman's number system and ternary mirror-symmetrical arithmetic. The Computer Journal, 2002, Vol. 45, No. 2: 222-236.
17. Stakhov A.P. The generalized golden sections and a new approach to geometric definition of a number. Ukrainian Mathematical Journal, Volume 56, 2004 (in Russian).
18. Stakhov A.P. Sacred Geometry and Harmony Mathematics. Academy of Trinitarism. Moscow: № 77-6567, publication 11176, 26.04.2004 (http://trinitas.ru/rus/doc/0202/010a/02020028.htm).
19. Stakhov A.P. Mathematics of Harmony as a new interdisciplinary direction of modern science. Academy of Trinitarism. Moscow: № 77-6567, publication 12371, 19.08.2005 (http://www.trinitas.ru/rus/doc/0232/004a/02320001.htm)
20. Stakhov A.P., Rozin B. On a new class of hyperbolic function. Chaos, Solitons & Fractals 2004, **23(2):** 379-389.
21. Stakhov A.P., Rozin B. The Golden Shofar . Chaos, Solitons & Fractals 2005, **26(3);** 677-684.
22. Stakhov A.P., Rozin B. The "golden" algebraic equations. Chaos, Solitons & Fractals 2005, **27 (5):** 1415-1421.
23. Stakhov A.P., Rozin B. Theory of Binet formulas for Fibonacci and Lucas *p*-numbers. Chaos, Solitons & Fractals 2005, 27 (5): 1162-1177.
24. Stakhov A.P., Rozin B. The continuous functions for the Fibonacci and Lucas *p*-numbers. Chaos, Solitons & Fractals 2006, 28 (4): 1014-1025.
25. Stakhov A.P. The Generalized Principle of the Golden Section and its applications in mathematics, science, and engineering. Chaos, Solitons & Fractals 2005, **26 (2)**: 263-289.
26. Stakhov A.P. Fundamentals of a new kind of Mathematics based on the Golden Section. Chaos, Solitons & Fractals 2005, 27 (5): 1124-1146.
27. Stakhov A.P. Fibonacci matrices, a generalization of the "Cassini formula," and a new coding theory. Chaos, Solitons & Fractals, 2006, Volume 30, Issue 1, 56-66.
28. Stakhov A. The "golden" matrices and a new kind of cryptography. Chaos, Solitons & Fractals 2007, Volume 32, Issue 3, 1138-1146.
29. Stakhov A. Rozin B. The Golden Section, Fibonacci series and new hyperbolic models of Nature. Visual Mathematics, Volume 8, No. 3, 2006 (http://members.tripod.com/vismath/pap.htm)
30. Kocer G., Tuglu N., Stakhov A. On the *m*-extension of the Fibonacci and Lucas *p*-numbers. Chaos, Solitons & Fractals (in Press).
31. Stakhov A.P. The golden section, sacred geometry, and harmony mathematics. Metaphysics: Century XXI (compiler and editor Y.S. Vladimirov). Moscow: "BINOM" publishing house, 2006, 174-215 (in Russian).
32. Stakhov A.P. Three "key" problems of mathematics on the stage of its origin, the "Harmony Mathematics" and its applications in contemporary mathematics, theoretical physics and computer science. Visual Mathematics, Volume 9, No.3, 2007 (http://members.tripod.com/vismath/pap.htm)
33. Stakhov A.P. Gazale formulas, a new class of the hyperbolic Fibonacci and Lucas functions and the improved method of the "golden" cryptography. Academy of Trinitarism, Moscow: № 77-6567, publication14098, 21.12.2006 (http://www.trinitas.ru/rus/doc/0232/004a/02321063.htm) (in Russian)
34. Stakhov A.P. Hyperbolic Fibonacci and Lucas functions: a history and applications. Academy of Trinitarism, Moscow, № 77-6567, publication 14429, 31.05.2007 (www.trinitas.ru/rus/doc/0232/009a/02321057.htm) (in Russian)
35. Stakhov A.P. "Strategic mistakes" in the mathematics development. Academy of Trinitarism, Moscow, № 77-6567, publication 14555, 27.08.2007 (www.trinitas.ru/rus/doc/0232/004a/02321070.htm) (in Russian)
36. Stakhov A.P. Three "Key" Problems of Mathematics on the Stage of its Origin and the "Harmony Mathematics" as Alternative Way of Mathematics Development. Mathematics & Design. Fifth International Mathematics & Design Conference – V M&D. Blumenau: Nova Letra, 2007: 88-102.
37. Stakhov A.P., Sluchenkova, A. Design of a New Coding Theory and Cryptography Based on the Fibonacci and "Golden" Matrices. Mathematics & Design. Fifth International Mathematics & Design Conference – V M&D. Blumenau: Nova Letra, 2007: 154-162.

38. Stakhov A.P. Three "key" problems of mathematics on the stage of its origin and new directions in the development of mathematics, theoretical physics and computer science. Moscow: № 77-6567, publication 14135, 12.01.2007 http://www.trinitas.ru/rus/doc/0232/004a/02321064.htm
39. Kline M.. The Loss of Certainty. New York. Oxford University Press. 1980.
40. Lebegue A. About measurement of magnitudes. Moscow: Publishing House "Uchpedgiz", 1960 (in Russian).
41. Kolmogorov A.N. Mathematics in its historical development. Moscow: Publishing House "Nauka," 1961 (in Russian).
42. Bodnar O.Y. The Golden Section and Non-Euclidean Geometry in Nature and Art. Lvov: Publishing House "Svit," 1994 (in Russian).
43. Klein F. Lectures on isosahedron and solutions to the 5th degree equation. Moscow: Nauka, 1989.
44. Bergman G. A number system with an irrational base. Mathematics Magazine, 1957, No 31: 98-119.
45. Spears C.P., Bicknell-Johnson, M. Asymmetric cell division: binomial identities for age analysis of mortal vs. immortal trees, Applications of Fibonacci Numbers, Vol. 7, 1998, 377-391.
46. Soroko E.M. Structural harmony of systems. Minsk: Publishing House "Nauka I technika," 1984 (in Russian).
47. de Spinadel V.W.. From the Golden Mean to Chaos. Nueva Libreria, 1998 (second edition, Nobuko, 2004).
48. Gazale M.J. Gnomon. From Pharaohs to Fractals. Princeton, New Jersey: Princeton University Press, 1999 (Russian translation, 2002).
49. Kappraff J. Beyond Measure. A Guided Tour Through Nature, Myth, and Number. Singapore, New Jersey, London, Hong Kong: World Scientific, 2002.
50. Hoggat V.E. Jr. Fibonacci and Lucas Numbers. - Boston, MA: Houghton Mifflin, 1969.
51. Gould H.W. A history of the Fibonacci *Q*-matrix and higher-dimensional problems. The Fibonacci Quarterly, 19, 1981, 250-257.
52. Brousentsov N.P. Computing machine "Setun" of Moscow State University. - New developments on computer technology. Kiev, Publishing House of the Kiev Cybernetics Institute, 1960 (in Russian).
53. Kharkevitch A.A. A struggle against noises. Moscow: Publishing House "Fizmatgiz," 1963 (in Russian).
54. Peterson W.W., Weldon, E.J. Error-correcting codes. Cambridge, Massachusetts, and London: The MIT Press, 1972.
55. Menezes A., Van Oorshot P.C. and Vanstone S.A. Handbook on Applied Cryptography. Boca Raton. Florida: CRC Press, 1996.
56. Gratia D. Quasi-crystals. Uspechi physicheskich nauk, 1988, Vol. 156: 347-363 (in Russian)
57. El Naschie M.S. On dimensions of Cantor set related systems. Chaos, Solitons & Fractals 1993; 3: 675-685.
58. El Naschie M.S. Quantum mechanics and the possibility of a Cantorian space-time. Chaos, Solitons & Fractals 1992; 1: 485-487.
59. El Naschie M.S. Is Quantum Space a Random Cantor Set with a Golden Mean Dimension at the Core? Chaos, Solitons & Fractals, 1994; 4(2); 177-179.
60. El Naschie M.S. Fredholm Operators and the Wave-Particle Duality in Cantorian Space. Chaos, Solitons & Fractals1998; 9(6): 975-978.
61. El Naschie M.S. On a class of general theories for high energy particle physics. Chaos, Solitons & Fractals 2002; 14: 649-668.
62. El Naschie M.S. On the topologic ground state of *E*-infinity spacetime and super string connection. Chaos, Solitons & Fractals, 2007; 32: 468-470.
63. El Naschie M.S. Feigenbaum scenario for turbulence and Cantorian E-infinity theory of high energy particle physics . Chaos, Solitons & Fractals, 2007, 32 (3), 911-915.
64. El Naschie M.S. Hilbert space, Poincaré dodecahedron and golden mean transfiniteness. Chaos, Solitons & Fractals, 2007, 31 (4), 787-793.
65. Petrunenko V.V. The Golden Section of quantum states and its astronomical and physical manifestations. Minsk: Publishing House "Pravo i economika," 2005 (in Russian).
66. Petoukhov S.V. Metaphysical aspects of the matrix analysis of genetic code and the golden section. Metaphysics: Century XXI. Moscow: "BINOM" publishing house, 2006, 216-250 (in Russian).
67. Statement by Professor Michael (Michal) Heller. At the Templeton Prize News Conference, March 12, 2008. http://www.templetonprize.org/bios.html